ALGEBRAIC SPACES AND QCOH SHEAVES

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These notes were for a talk I gave in a graduate student seminar on Algebraic Spaces and Stacks. Any mistakes are my own.

Brief Recap

Here we lay out a few fundamental details of algebraic spaces which will be of use later on. Specifically, we spell out again the connection between the formal definition of algebraic spaces (as an etale sheaf over $\mathrm{Spec}(S)$ for which the diagonal map is representable and which admits an etale surjection over S from a scheme U), and the informal definition as the etale sheaf one gets by sheafifying the presheaf which takes an etale equivalence relation R on a scheme X and for every scheme $T \to S$ gives $X(T)/R(T)$.

Indeed, one can check that the latter sheaf has representable diagonal and naturally has an etale surjection from X. Given the former version of an algebraic space X/S, we take U the scheme which admits an etale surjection to X as stipulated, and form $U \times_X U \hookrightarrow X \times_S X$. One then checks that $R = U \times_X U$ is an etale equivalence relation, and the sheaf associated to the quotient presheaf as above gives an isomorphic etale sheaf over S.

Definitions and Basic Properties

As with the classical etale topology of schemes, we begin by defining the small etale site.

Let X be an algebraic space (if no base is specified, assume we are working over $\mathbb{Z}$). We define the small etale site by specifying that the objects are etale morphisms of algebraic spaces $Y \to X$, and covers are appropriately defined as they were for schemes. Note that covers are automatically etale. We denote this site by $\mathrm{Et}(X)$, and the sheaves on this site by X_{et}.

We can, of course, restrict the objects so that Y is a scheme. We then look at this subcategory. Since covers can be refined so that the domain consists of schemes, we can give this subcategory (call it $\mathrm{Et}'(X)$) the induced topology. Then, we find that since we have an inclusion of sites, $\mathrm{Et}'(X) \subset \mathrm{Et}(X)$, we have an induced map of topoi $X_{et} \to X_{et'}$.

For formal reasons, this map of topoi is an equivalence. But we can use this to easily define, for example, a structure sheaf in X_{et} by first defining it on $X_{et'}$ via $(Y \to X) \mapsto \Gamma(Y_{Zar}, \mathcal{O}_Y)$, where Y is a scheme.

So, let's say we have an algebraic space X with etale surjection from a scheme $U \to X$. Let $R = U \times_X U$, and let $R' = U \times_x U \times_X U$ be the triple intersection.

We essentially want to describe etale sheaves on X as etale sheaves on U which glue appropriately when we quotient out by the equivalence relation R. Descent allows us to do this very cleanly. Namely, letting $\mathrm{pr}_{ij} : R' \to R$ be the projections onto the (i, j)-factors, and,

$$\begin{tikzcd} R \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

we denote by $(R \rightrightarrows U)_{et}$ the category consisting of (F_U, ϵ) where F_U is an etale sheaf on U, and

$$\epsilon : s^*F_U \to t^*F_U$$

is an isomorphism, and this isomorphism agrees on triple intersections via commutativity of the big diagram,

$$\begin{tikzcd} (s \circ \mathrm{pr}_{12})^*F_U \arrow[r, "\mathrm{pr}_{12}^*\epsilon"] \arrow[d, "\cong"'] & (t \circ \mathrm{pr}_{12})^*F_U \arrow[r, "\cong"] & (s \circ \mathrm{pr}_{23})^*F_U \arrow[r, "\mathrm{pr}_{23}^*\epsilon"] & (t \circ \mathrm{pr}_{23})^*F_U \arrow[d, "\cong"] \\ (s \circ \mathrm{pr}_{13})^*F_U \arrow[rrr, "\mathrm{pr}_{13}^*\epsilon"'] & & & (t \circ \mathrm{pr}_{13})^*F_U \end{tikzcd}$$

There is a natural functor $X_{et} \to (R \rightrightarrows U)_{et}$, and for very general reasons, this is an equivalence of categories (in general, one needs only that finite limits are representable in $\mathrm{Et}(X)$).

If, in the definition of $(R \rightrightarrows U)_{et}$ we insisted that F_U would be a sheaf of $\mathcal{O}_U$-modules, and ϵ would be an isomorphism of $\mathcal{O}_{U\times_X U}$-modules which satisfied the appropriate cocycle condition on triple intersections, then we would get an equivalence between the category of $\mathcal{O}_X$-modules and the modified $(R \rightrightarrows U)_{et}$.

We finally get to the big definition of the talk,

Definition: $\mathcal{F} \in \mathcal{O}_X$-mod is quasi-coherent if the associated sheaf in $(R \rightrightarrows U)_{et}$ is a quasicoherent sheaf of $\mathcal{O}_U$-modules.

We say $\mathcal{F}$ is coherent if the associated sheaf on U is coherent.

Many of the statements you could make for (quasi-)coherent sheaves on schemes are true for (quasi-)coherent sheaves on algebraic spaces. In this talk, we note some of the big hits.

First, we note that there is a natural pushforward of $\mathcal{O}_X$-modules, f_*, and it admits a left-adjoint pullback, f^*. The pullback of a quasicoherent sheaf is quasicoherent, and the pushforward under a quasi-compact and quasi-separated morphism of algebraic spaces preserves quasicoherence.

We can define a form of relative Spec for algebraic spaces as follows: for $\mathfrak{A}$ a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, we define the functor $\mathrm{Spec}_X(\mathfrak{A})$ from $(\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Set}$ via,

$$T \mapsto (f : T \to X, \epsilon : f^*\mathfrak{A} \to \mathcal{O}_T)$$

This functor carries a natural forgetful transformation,

$$\pi : \mathrm{Spec}_X(\mathfrak{A}) \to X$$

which for a scheme T sends the pair (f, ϵ) to f (for X a scheme, this is the functor of points, and for algebraic spaces, this takes the image of the identity map on T).

It turns out that $\mathrm{Spec}_X(\mathfrak{A})$ is an algebraic space, and π is an affine morphism.

Furthermore, for any affine morphism of algebraic spaces $g : Y \to X$, we have that $g_*\mathcal{O}_Y$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, and this gives an (anti-)equivalence of categories between quasi-coherent sheaves of $\mathcal{O}_X$-algebras and affine morphisms to X. For details of $\mathrm{Spec}_X(\mathfrak{A})$ being an algebraic space and the map π being an affine morphism, refer to Olsson's text.

Now that we have a clean understanding of affine morphisms of algebraic spaces, we can define some of the concepts we have for schemes.

Analogues of Scheme-theoretic Concepts

Definition:(Scheme-theoretic Image) For $f : X \to Y$ a qcqs (quasi-compact and quasiseparated) morphism of algebraic spaces, if K is the kernel of the induced map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$, then $\mathcal{O}_Y/K$ is a quasicoherent sheaf of $\mathcal{O}_Y$-algebras, and so we can define the relative Spec, $\mathrm{Spec}_Y(\mathcal{O}_Y/K)$, which will be our scheme-theoretic image.

We can similarly define the maximal reduced subspace as the relative Spec of $\mathcal{O}_X/\mathcal{N}_X$ where $\mathcal{N}_X$ is the sheaf of nilpotents.

We also have an equivalence between closed subspaces (up to isomorphism) of X and quasi-coherent sheaves of ideals $\mathcal{J} \subset \mathcal{O}_X$ in much the same way as we did for schemes.

Stein Factorization

Recall that for schemes, if $f : X \to Y$ is qcqs, f is the composition,

$$X \xrightarrow{a} X' \xrightarrow{b} Y$$

where b is affine and a is Stein (i.e. the sheaf map $\mathcal{O}_{X'} \to a_*\mathcal{O}_X$ is an isomorphism of sheaves). The way we get this factorization is by letting $X' = \mathrm{Spec}_Y(f_*\mathcal{O}_X)$.

Since we have relative Spec for algebraic spaces as well, we can give Stein factorizations for certain morphisms $f : X \to Y$ of algebraic spaces. The morphisms must be separated and quasi-finite (i.e. locally quasi-finite and finite type). Moreover, in that case, the associated a we get will be an open embedding. We leave out the proof because it is fairly long.

Note, however, that the algebraic space we factor through need not be as nice as either X or Y. Indeed, simply because Y is Noetherian does not mean that X' needs to be Noetherian. This is tantamount to showing that the global sections of X need not be Noetherian (see example of Osserman, de Jong, Conrad, and Vakil).

The speaker will say a few words about nilpotent thickenings.

There is also some confusion about the exact statement of Chow's lemma, which is not so clear to the speaker. Roughly, if $f : X \to S$ is a separated morphism of finite type where X is a reduced algebraic space and S is a Noetherian scheme, then we can find a scheme X' and a proper birational morphism $g : X' \to X$ such that X' is quasi-projective over S. Because $X' \to X$ is proper, the induced map into $\mathbb{P}^n_X$ is a closed immersion.

Cohomology

We can define cohomology of quasicoherent sheaves on algebraic spaces in much the same way as we do for schemes. Concretely, this means that the category of quasicoherent sheaves on X is an abelian category with enough injectives.

The big Kahuna for us that we want to get to in this talk is the following generalization of a classic theorem in scheme theory:

Theorem: If $f : X \to Y$ is a proper morphism of locally Noetherian algebraic spaces, and $\mathcal{F}$ is a coherent sheaf on X, then $R^q f_* F$ is a coherent sheaf on Y for all q.

The proof will essentially be the same as in the scheme world, but this proof is not commonly presented in a course on schemes, so we will present it here for algebraic spaces. The proof uses a standard devissage argument.

Devissage: Let $f : X \to S$ be a qcqs morphism of algebraic spaces with S an affine scheme and both spaces locally Noetherian. Let $\mathcal{A} \subset \mathrm{Coh}(X)$ be a full subcategory. If,
(i) The 0 sheaf is in $\mathcal{A}$
(ii) For any exact sequence of sheaves in $\mathrm{Coh}(X)$, if any two are in $\mathcal{A}$, then so is the third.
(iii) For every integral closed subspace $Z \hookrightarrow X$, there is some $\mathcal{G} \in \mathcal{A}$ such that $\mathrm{supp}(\mathcal{G}) = Z$
(iv) $\mathcal{F} \oplus \mathcal{G} \in \mathcal{A}$ implies $\mathcal{F}, \mathcal{G} \in \mathcal{A}$
Then, $\mathcal{A} = \mathrm{Coh}(X)$.

We leave the proof of this fact out. But now, we are in a position to prove the theorem above.

Note that we can reduce to the case Y is an affine scheme. Our strategy will be to consider $\mathcal{A} := \{\mathcal{F} \in \mathrm{Coh}(X) | R^q f_* \mathcal{F} \in \mathrm{Coh}(Y) \text{ for all } q \geq 0\}$.

We will show that $\mathcal{A}$ satisfies the conditions of the devissage argument and conclude.

The only difficult step is proving (iii), i.e. for any integral closed subspace $Z \hookrightarrow X$, there is an $\mathcal{G}$ such that $\mathrm{supp}(\mathcal{G}) = Z$. We can reduce to the case $Z = X$, i.e. we are tasked with finding a coherent sheaf $\mathcal{G} \in \mathcal{A}$ such that $\mathrm{supp}(\mathcal{G}) = X$.

To do this, we use Chow's lemma to get $\pi : X' \to X$ a proper birational map such that the composition $h : X' \to S$ is projective. Note that π itself will be projective, so we can look at $\mathcal{G} = \pi_* \mathcal{O}_{X'}(n)$, as it will be coherent. Now, note that $\mathcal{O}_{X'}(n)$ does not vanish at the generic point of X', and π is birational, so $\mathcal{G}$ does not vanish at the generic point of X. Also, $R^i h_* \mathcal{O}_{X'}(n)$ is coherent for all nonnegative q.

So, we can use a spectral sequence (probably the Leray s.s.) to get,

$$R^p f_*(R^q \pi_* \mathcal{O}_{X'}(n)) \implies R^{p+q} h_* \mathcal{O}_{X'}(n)$$

Since π is projective, for sufficiently large n, we get that $R^{\geq 1}\pi_*\mathcal{O}_{X'}(n) = 0$. Thus, by taking n sufficiently large, our spectral sequence becomes,

$$R^p f_*(\pi_*\mathcal{O}_{X'}(n)) = R^p f_*\mathcal{G} = R^p h_*\mathcal{O}_{X'}(n)$$

and hence $\mathcal{G} \in \mathcal{A}$ (since $R^p h_*\mathcal{O}_{X'}(n)$ is coherent for all p) as required. This completes the proof.

REFERENCES (INFORMAL)

The references are the same as those listed on the website for the seminar, with the relevant sections used.